

Deformation of quantum oscillator and of its interaction with environment

A. Isar^{†‡(a)} and W. Scheid [‡]

[†]*Department of Theoretical Physics, Institute of Physics and Nuclear Engineering
Bucharest-Magurele, Romania*

[‡]*Institut für Theoretische Physik der Justus-Liebig-Universität
Giessen, Germany*

Abstract

A master equation for the deformed quantum harmonic oscillator interacting with a dissipative environment, in particular with a thermal bath, is derived in the microscopic model by using perturbation theory, for the case when the interaction is deformed. The coefficients of the master equation and of equations of motion for observables depend on the deformation function. The steady state solution of the equation for the density matrix in the number representation is obtained and shown that it satisfies the detailed balance condition. The equilibrium energy of the deformed harmonic oscillator, calculated in the approximation of small deformation, does not depend on the deformation of interaction operators.

PACS numbers: 03.65.Bz, 05.30.-d, 05.40.+j, 02.20.Sv

(a) e-mail address: isar@theory.nipne.ro

1 Introduction

For more than a decade a constant interest has been manifested to the study of deformations of Lie algebras, so-called quantum algebras or quantum groups, whose rich structure produced important results and consequences in statistical mechanics, quantum field theory, conformal field theory, quantum and nonlinear optics, nuclear and molecular physics [1, 2]. Their use in physics became stronger with the introduction of the q -deformed Heisenberg-Weyl algebra (q -deformed quantum harmonic oscillator) by Biedenharn [3] and MacFarlane [4] in 1989. There are, at least, two properties which make q -oscillators interesting objects for physics. The first is the fact that they naturally appear as the basic building blocks of completely integrable theories. The second concerns the connection between q -deformation and nonlinearity. In Refs. [5, 6, 7] it was shown that the q -oscillator leads to nonlinear vibrations with a special kind of the dependence of the frequency on the amplitude. The q -deformed Bose distribution has been obtained and it was also shown how q -nonlinearity produces a correction to the Planck distribution formula [5, 6, 8].

The present paper is the third one in the series of papers devoted to the study of the influence of quantum deformation on quantum dissipation. In Ref. [9], using a variant of Mancini's model [10], we derived a master equation for the f -deformed oscillator in the presence of a dissipative environment for an undeformed interaction between system and environment. Then in Ref. [11] we obtained a Lindblad master equation for the ordinary harmonic oscillator interacting with an environment through a deformed interaction. The equations of motion obtained for different observables had a strong dependence on the deformation. In the present paper we set a master equation for the deformed harmonic oscillator in the presence of a dissipative environment, for the case of a deformed interaction of the system with its environment. This equation is shown to be a deformed version of the master equation obtained in the framework of the Lindblad theory for open quantum systems [12]. When the deformation becomes zero, we recover the Lindblad master equation for the damped harmonic oscillator [13, 14]. We are interested in studying the role of nonlinearities which appear in the master equation. This goal is motivated by the fact that the q -oscillator can be considered as a physical system with a specific nonlinearity, called q -nonlinearity [5, 6]. For a certain choice of the environment coefficients, a master equation for the damped deformed oscillator has also been derived by Mancini [10].

The paper is organized as follows. In Sec. 2 we remind the basics of the f -deformed quantum oscillator, in particular of the q -oscillator. In Sec. 3 we derive a master equation for the f -deformed oscillator in the presence of a dissipative environment, for the case of a deformed interaction between system and environment. The equations of motion obtained for different observables present a strong dependence on the deformation. Then in Sec. 4 we give the equation for the density matrix in the number representation and find the stationary state. In the particular case when the environment is a thermal bath we obtain an expression for the equilibrium energy of the oscillator in the approximation of a small deformation parameter. A summary and conclusions are given in Sec. 5.

2 f - and q -deformed quantum oscillators

It is known that the ordinary operators $\{1, a, a^\dagger, N\}$ form the Lie algebra of the Heisenberg-Weyl group and the linear harmonic oscillator can be connected with the generators of the Heisenberg-Weyl Lie group. The f -deformed quantum oscillators [15] are defined by the algebra generated by the operators $\{1, A, A^\dagger, N\}$, where the Hermitian number operator N is not equal to $A^\dagger A$ as in the ordinary case.

The f -deformed oscillator operators are given as follows [15]:

$$A = af(N) = f(N+1)a, \quad A^\dagger = f(N)a^\dagger = a^\dagger f(N+1), \quad (1)$$

where $N = a^\dagger a$. They satisfy the commutation relations

$$[A, N] = A, \quad [A^\dagger, N] = -A^\dagger \quad (2)$$

and

$$[A, A^\dagger] = (N+1)f^2(N+1) - Nf^2(N). \quad (3)$$

The function f , which is a characteristics for the deformation, has a dependence on a deformation parameter α such that when the deformation disappears, then $f(N, \alpha = 0) = 1$ and the usual algebra is recovered. Transformation (1) of the operators a, a^\dagger to A, A^\dagger represents a nonlinear noncanonical transformation, since it does not preserve the commutation relation, i. e. $[A, A^\dagger] \neq 1$.

The notion of f -oscillators generalizes the notion of q -oscillators ($q = 1 + \alpha$). Indeed, for

$$f(N) = \sqrt{\frac{[N]}{N}}, \quad [N] \equiv \frac{q^N - q^{-N}}{q - q^{-1}}, \quad (4)$$

where q is the deformation parameter (a dimensionless c -number), the operators A and A^\dagger in Eq. (2) become the q -deformed boson annihilation and creation operators [3, 4]. For the q -deformed harmonic oscillator the commutation relation (3) becomes

$$[A, A^\dagger] = [N+1] - [N]. \quad (5)$$

If q is real positive, then

$$[N] = \frac{\sinh(N \ln q)}{\sinh(\ln q)} \quad (6)$$

and the condition of Hermitian conjugation $(A^\dagger)^\dagger = A$ is satisfied. In the limit $q \rightarrow 1$, q -deformed operators tend to the ordinary operators because $\lim_{q \rightarrow 1} [N] = N$. Then Eq. (5) goes to the usual boson commutation relation $[A, A^\dagger] = 1$.

The q -deformed boson operators A and A^\dagger can be expressed in terms of the usual boson operators a and a^\dagger (satisfying $[a, a^\dagger] = 1$, $N = a^\dagger a$ and $[a, N] = a$, $[a^\dagger, N] = -a^\dagger$) through the relations [16, 17] (see Eqs. (1)):

$$A = a\sqrt{\frac{[N]}{N}} = \sqrt{\frac{[N+1]}{N+1}}a, \quad A^\dagger = \sqrt{\frac{[N]}{N}}a^\dagger = a^\dagger\sqrt{\frac{[N+1]}{N+1}}. \quad (7)$$

Using the nonlinear map (4) [18, 19], the q -oscillator has been interpreted [5, 6] as a nonlinear oscillator with a special type of nonlinearity which classically corresponds to an energy dependence of the oscillator frequency. Other nonlinearities can also be introduced by making the frequency to depend on other constants of motion, different from energy [5, 15], through the deformation function f . Other examples of deformation functions f can be found in Refs. [20, 21, 22, 23].

The Hamiltonian of the f -deformed harmonic oscillator (ω is the ordinary frequency) is a function of N :

$$\mathcal{H} = \frac{\hbar\omega}{2}(AA^\dagger + A^\dagger A) = \frac{\hbar\omega}{2} \left((N+1)f^2(N+1) + Nf^2(N) \right). \quad (8)$$

It is diagonal on the eigenstates $|n\rangle$ in the Fock space and its eigenvalues are

$$E_n = \frac{\hbar\omega}{2} \left((n+1)f^2(n+1) + nf^2(n) \right). \quad (9)$$

In the limit $f \rightarrow 1$ ($q \rightarrow 1$ for q -oscillators), we recover the ordinary expression $E_n = \hbar\omega(n + 1/2)$.

Using the operator Heisenberg equation with Hamiltonian (8)

$$i\hbar \frac{dA}{dt} = [A, \mathcal{H}] \quad (10)$$

we obtain the following solutions to the Heisenberg equations of motion for the f -deformed operators A and A^\dagger defined in Eqs. (1) [10, 15]:

$$A(t) = \exp(-i\omega\Omega(N)t) A, \quad A^\dagger(t) = A^\dagger \exp(i\omega\Omega(N)t), \quad (11)$$

where $\Omega(N)$ is the operator defined as

$$\Omega(N) = \frac{1}{2} \left((N+2)f^2(N+2) - Nf^2(N) \right). \quad (12)$$

For a q -deformed harmonic oscillator,

$$\Omega(N) = \frac{1}{2}([N+2] - [N]) \quad (13)$$

and for a small deformation parameter τ ($\tau = \ln q$),

$$\Omega(N) = 1 + \frac{\tau^2}{2}(N+1)^2. \quad (14)$$

3 Quantum Markovian master equation

In order to discuss the dynamics of the open systems S, we use a microscopic description of the composite system. As the subsystem S of interest we take the f -deformed harmonic oscillator with Hamiltonian \mathcal{H} (8) and the environment R (reservoir, bath) with the Hamiltonian H_R . The coupled system S+R with the total Hamiltonian $H_T = \mathcal{H} + H_R + \mathcal{V}$ (\mathcal{V} is the f -deformed interaction Hamiltonian) is described by a density operator $\chi(t)$, which evolves in time according to the von Neumann-Liouville equation

$$\frac{d\chi(t)}{dt} = -\frac{i}{\hbar}[H_T, \chi(t)]. \quad (15)$$

When the Hamiltonian evolution of the total system is projected onto the space of the harmonic oscillator, the reduced density operator of the subsystem is given by $\rho(t) = \text{Tr}_R \chi(t)$. The derivation of the reduced density operator in which the operators of the environment system have been eliminated up to second order of the perturbation theory can be taken from literature [24, 25, 26, 27, 28]. Following [27] and the procedure used in the previous paper [9], we point the main steps in obtaining the master equation which describes the time evolution of the damped deformed harmonic oscillator. We assume that the initial state R_0 of the environment at $t = 0$ does not depend on the state $\rho(0)$ of the subsystem. Then $\chi(0) = \rho(0)R_0$. At later times correlations between S and R arise due to the coupling of the system and environment through \mathcal{V} . However, we assume that this coupling is weak and then at any instant of time, not only for the initial time, the reservoir and the oscillator are approximately decoupled. Furthermore, R is a large system whose state should be virtually unaffected by its coupling to S. We then write $\chi(t) = \rho(t)R_0$. In the previous paper [9], we considered the interaction potential of the linear form in the coordinate and momentum operators in the Hilbert space of the subsystem S. In analogy to that model, we assume in the present paper a f -deformed interaction potential \mathcal{V} of the form

$$\mathcal{V} = \sum_{i=1,2} S_i \Gamma_i, \quad (16)$$

where $S_1 = Q$ and $S_2 = P$ are the f -deformed coordinate and momentum operators in the Hilbert space of the subsystem S (m is the oscillator mass):

$$Q = \sqrt{\frac{\hbar}{2m\omega}}(A^\dagger + A), \quad P = i\sqrt{\frac{\hbar m\omega}{2}}(A^\dagger - A), \quad (17)$$

with A^\dagger, A defined in Eqs. (1) and Γ_i are Hermitian operators in the Hilbert space of the environment. Then we obtain the following master equation for the density operator of the

open quantum system in the Born-Markov approximation:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[\mathcal{H}, \rho(t)] + \frac{1}{\hbar^2} \sum_{i,j=1,2} \int_0^t dt' \{C_{ij}^*(t') [S_i, \rho(t) S_j(-t')] + C_{ij}(t') [S_j(-t') \rho(t), S_i]\}, \quad (18)$$

where the coefficients $C_{ij}(t') = \text{Tr}_R\{R_0 \Gamma_i(t') \Gamma_j\}$ are correlation functions of the environment operators. It is assumed that the correlation functions decay very rapidly on the time scale on which $\rho(t)$ varies. Ideally, we might take $C_{ij}(t') \sim \delta(t')$. The Markov approximation relies on the existence of two widely separated time scales: a slow time scale for the dynamics of the system S and a fast time scale characterizing the decay of environment correlation functions [27].

In order to get the time dependence of the operators of coordinate $S_1(t) = Q(t)$ and momentum $S_2(t) = P(t)$ used in Eq. (18), we express them through the relations

$$Q(t) = \sqrt{\frac{\hbar}{2m\omega}} (A^\dagger(t) + A(t)), \quad P(t) = i\sqrt{\frac{\hbar m\omega}{2}} (A^\dagger(t) - A(t)) \quad (19)$$

and then insert Eqs. (11) for $A(t)$ and $A^\dagger(t)$. Then the master equation (18) takes the following form:

$$\begin{aligned} \frac{d\rho(t)}{dt} = & -\frac{i}{\hbar}[\mathcal{H}, \rho(t)] \\ & + \frac{1}{2\hbar^2} \int_0^t dt' (C_{11}^*(t') [Q, \rho(t) \left(QE_- + E_+ Q - \frac{i}{m\omega} (PE_- - E_+ P) \right)] \\ & \quad + C_{11}(t') \left[\left(QE_- + E_+ Q - \frac{i}{m\omega} (PE_- - E_+ P) \right) \rho(t), Q \right] \\ & \quad + iC_{22}^*(t') [P, \rho(t) (m\omega (QE_- - E_+ Q) - i(PE_- + E_+ P))] \\ & \quad + iC_{22}(t') [(m\omega (QE_- - E_+ Q) - i(PE_- + E_+ P)) \rho(t), P] \\ & \quad + iC_{12}^*(t') [Q, \rho(t) (m\omega (QE_- - E_+ Q) - i(PE_- + E_+ P))] \\ & \quad + C_{21}(t') \left[\left(QE_- + E_+ Q - \frac{i}{m\omega} (PE_- - E_+ P) \right) \rho(t), P \right] \\ & \quad + C_{21}^*(t') [P, \rho(t) \left(QE_- + E_+ Q - \frac{i}{m\omega} (PE_- - E_+ P) \right)] \\ & \quad + iC_{12}(t') [(m\omega (QE_- - E_+ Q) - i(PE_- + E_+ P)) \rho(t), Q]), \end{aligned} \quad (20)$$

where we have introduced the following notations:

$$E_+ = \exp(i\omega\Omega(N)t'), \quad E_- = \exp(-i\omega\Omega(N)t'). \quad (21)$$

If the environment is sufficiently large we may assume that the time correlation functions decay fast enough to zero for times t' longer than the relaxation time t_B of the environment: $t' \gg t_B$. Therefore, if we are interested in the dynamics of the subsystem over times which are

longer than the environment relaxation time, $t \gg t_B$, we may use the Markov approximation and replace the upper limit of integration t by ∞ . Physically, this amounts to assuming that the memory functions $C_{ij}(t')E_{\pm}(t')$ decay over a time which is much shorter than the characteristic evolution time of the system of interest. We evaluate the integrals in Eq. (20) like in the preceding paper [9]. After certain assumptions [24, 28], one can define the complex decay rates, which govern the rate of relaxation of the system density operator, as follows:

$$\int_0^{\infty} dt' C_{11}(t') E_{\pm} = \int_0^{\infty} dt' C_{11}^*(t') E_{\pm} = D_{pp}(\Omega), \quad (22)$$

$$\int_0^{\infty} dt' C_{22}(t') E_{\pm} = \int_0^{\infty} dt' C_{22}^*(t') E_{\pm} = D_{qq}(\Omega), \quad (23)$$

$$\int_0^{\infty} dt' C_{12}(t') E_{\pm} = \int_0^{\infty} dt' C_{21}^*(t') E_{\pm} = -D_{pq}(\Omega) + \frac{i\hbar}{2}\lambda(\Omega). \quad (24)$$

In fact, $D_{pp}(\Omega)$, $D_{qq}(\Omega)$, $D_{pq}(\Omega)$ and $\lambda(\Omega)$ play the role of deformed diffusion and, respectively, dissipation coefficients. The existence of these coefficients reflects the fact that, due to the interaction, the energy of the system is dissipated into the environment, but noise arises also (in particular, thermal noise), since the environment also distributes some of its energy back to the system. Then the master equation (20) for the damped deformed harmonic oscillator takes the form:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[\mathcal{H}, \rho] \\ & + \frac{1}{2\hbar^2} \left(\left(\{D_{pp}(\Omega), Q\} + \frac{i}{m\omega}[D_{pp}(\Omega), P] \right) \rho, Q \right) + \left(\{D_{qq}(\Omega), P\} - im\omega[D_{qq}(\Omega), Q] \right) \rho, P \\ & + \left(m\omega[iD_{pq}(\Omega) + \frac{\hbar}{2}\lambda(\Omega), Q] - \{D_{pq}(\Omega) - \frac{i\hbar}{2}\lambda(\Omega), P\} \right) \rho, Q \\ & - \left(\frac{1}{m\omega}[iD_{pq}(\Omega) - \frac{\hbar}{2}\lambda(\Omega), P] + \{D_{pq}(\Omega) + \frac{i\hbar}{2}\lambda(\Omega), Q\} \right) \rho, P + H.c. \end{aligned} \quad (25)$$

Here the curly parentheses mean anticommutators. We notice that the deformation is present in both the commutator containing the oscillator Hamiltonian \mathcal{H} , as well as in the dissipative part of the master equation, which describes the influence of the environment on the deformed oscillator. This master equation preserves the Hermiticity property of the density operator and the normalization (unit trace) at all times, if at the initial time it has these properties. In the limit $f \rightarrow 1$ ($\Omega \rightarrow 1$), the deformation disappears and Eq. (25) becomes the Markovian master equation for the damped harmonic oscillator, obtained in the Lindblad theory for open quantum systems, based on completely positive dynamical

semigroups [12, 13, 14]. The fact that we introduced a f -deformed interaction between the open system and environment is reflected in the presence of deformed operators P and Q in the dissipative part of Eqs. (20) and (25), along with the deformed diffusion operator coefficients, compared to the previous model [9, 10], where the dissipative part of the master equation contains the deformed diffusion coefficients and the usual operators p and q .

Expressing the coordinate and momentum operators back in terms of the creation and annihilation operators, introducing the notations

$$D_+(\Omega) \equiv \frac{1}{2\hbar} \left(m\omega D_{qq}(\Omega) + \frac{D_{pp}(\Omega)}{m\omega} \right), \quad D_-(\Omega) \equiv \frac{1}{2\hbar} \left(m\omega D_{qq}(\Omega) - \frac{D_{pp}(\Omega)}{m\omega} \right) \quad (26)$$

and assuming, like in Ref. [9], that $\lambda(\Omega) = \lambda = \text{const}$, the master equation (25) for the damped deformed harmonic oscillator takes the form

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[\mathcal{H}, \rho] + ([D_+(\Omega)af(N), \rho], f(N)a^\dagger] \\ & - [[f(N)a^\dagger \left(D_-(\Omega) + \frac{i}{\hbar}D_{pq}(\Omega) \right), \rho], f(N)a^\dagger] - \frac{\lambda}{2}[f(N)a^\dagger, \{af(N), \rho\}] + H.c., \end{aligned} \quad (27)$$

with \mathcal{H} given by Eq. (8).

In the particular case of a thermal equilibrium of the bath at temperature T (k is the Boltzmann constant), we take the diffusion coefficients of the form (in concordance with results of Refs. [9, 10])

$$m\omega D_{qq}(\Omega) = \frac{D_{pp}(\Omega)}{m\omega} = \frac{\hbar}{2}\lambda \coth \frac{\hbar\omega\Omega}{2kT}, \quad D_{pq}(\Omega) = 0 \quad (28)$$

and the master equation (27) takes the form

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\mathcal{H}, \rho] + \frac{\lambda}{2} \left([[\coth \frac{\hbar\omega\Omega}{2kT}af(N), \rho], f(N)a^\dagger] - [f(N)a^\dagger, \{af(N), \rho\}] + H.c. \right). \quad (29)$$

In the limit $\Omega \rightarrow 1$, the deformed diffusion coefficients (28) take the known form obtained for the damped harmonic oscillator, if the asymptotic state is a Gibbs state [13, 14]:

$$D_{pp} = \frac{\hbar m\omega}{2}\lambda \coth \frac{\hbar\omega}{2kT}, \quad D_{qq} = \frac{\hbar}{2m\omega}\lambda \coth \frac{\hbar\omega}{2kT}, \quad D_{pq} = 0. \quad (30)$$

If the bath temperature is $T = 0$, the master equation (29) simplifies:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i\omega}{2}[(N+1)f^2(N+1) + Nf^2(N), \rho] \\ & -\lambda \left(Nf^2(N)\rho + \rho Nf^2(N) - 2f(N+1)a\rho a^\dagger f(N+1) \right). \end{aligned} \quad (31)$$

The meaning of the master equation (29) becomes clear when we transform it into equations satisfied by the expectation values of observables involved in the master equation,

$\langle O \rangle = \text{Tr}[\rho(t)O]$, where O is the operator corresponding to such an observable. We give an example, multiplying both sides of Eq. (29) by the number operator N and taking the trace. Then we obtain the following equation of motion for the expectation value of N :

$$\begin{aligned} \frac{d}{dt} \langle N \rangle = & \lambda \left(\langle \left(\coth \frac{\hbar\omega\Omega(N)}{2kT} - 1 \right) (N+1) f^2(N+1) \rangle \right. \\ & \left. - \langle \left(\coth \frac{\hbar\omega\Omega(N-1)}{2kT} + 1 \right) N f^2(N) \rangle \right). \end{aligned} \quad (32)$$

This equation leads to a time dependence of the averaged number of quanta on dissipation, temperature and deformation, compared to the case of an oscillator without dissipation, where the expectation number of quanta is conserved. In the case of a q -deformation, Eq. (32) takes the form

$$\frac{d}{dt} \langle N \rangle = \lambda \left(\langle \left(\coth \frac{\hbar\omega\Omega(N)}{2kT} - 1 \right) [N+1] \rangle - \langle \left(\coth \frac{\hbar\omega\Omega(N-1)}{2kT} + 1 \right) [N] \rangle \right). \quad (33)$$

If, in addition, the temperature of the thermal bath is $T = 0$, Eq. (33) becomes

$$\frac{d}{dt} \langle N \rangle = -2\lambda \langle [N] \rangle. \quad (34)$$

In order to obtain an approximate solution of this equation, we work in the limit of a small deformation parameter $\tau = \ln q$ (q real). Then we can take [1, 2]

$$[N] = N + \frac{\tau^2}{6}(N^3 - N) \quad (35)$$

and, making also the assumption $\langle N^3 \rangle \approx \langle N \rangle^3$, Eq. (34) reduces to the following differential equation:

$$\frac{d}{dt} \langle N \rangle = -2\lambda \left(\langle N \rangle + \frac{\tau^2}{6} (\langle N \rangle^3 - \langle N \rangle) \right). \quad (36)$$

By integrating this equation we obtain

$$\frac{\langle N(t) \rangle}{\sqrt{1 - \frac{\tau^2}{6} + \frac{\tau^2}{6} \langle N(t) \rangle^2}} = \frac{\langle N(0) \rangle e^{-2\lambda(1 - \frac{\tau^2}{6})t}}{\sqrt{1 - \frac{\tau^2}{6} + \frac{\tau^2}{6} \langle N(0) \rangle^2}}, \quad (37)$$

from where we can obtain the following expression for the expectation value of the number operator in the approximation of a small deformation parameter:

$$\begin{aligned} \langle N(t) \rangle = & \frac{\langle N(0) \rangle \sqrt{1 - \frac{\tau^2}{6}} e^{-2\lambda(1 - \frac{\tau^2}{6})t}}{\sqrt{1 - \frac{\tau^2}{6} + \frac{\tau^2}{6} \langle N(0) \rangle^2} (1 - e^{-4\lambda(1 - \frac{\tau^2}{6})t})} \\ \approx & \langle N(0) \rangle e^{-2\lambda(1 - \frac{\tau^2}{6})t} \left(1 - \frac{\tau^2}{12} \langle N(0) \rangle^2 (1 - e^{-4\lambda(1 - \frac{\tau^2}{6})t}) \right). \end{aligned} \quad (38)$$

In the limit $\tau \rightarrow 0$, we obtain

$$\langle N(t) \rangle = \langle N(0) \rangle e^{-2\lambda t}, \quad (39)$$

which is the expression of the expectation value of the number operator of the damped harmonic oscillator obtained in the Lindblad theory for open quantum systems.

We consider another example, taking the simplest case of a thermal bath at $T = 0$, when both diffusion and dissipation coefficients do not depend on the deformation, $D_+ = \lambda/2 = \text{const}$. Even in this situation, the equations of motion for the expectation values are yet complicated, because they do not form a closed system. Multiplying both sides of Eq. (31) by the operator a and, respectively, $\Omega(N)a$ and taking throughout the trace, we get the following equations for the expectation values of these operators:

$$\begin{aligned} \frac{d}{dt} \langle a \rangle &= -i\omega \langle \Omega(N)a \rangle \\ -\lambda \langle \left((N+1)f^2(N+1) + Nf^2(N) - 2Nf(N)f(N+1) \right) a \rangle, \end{aligned} \quad (40)$$

$$\begin{aligned} \frac{d}{dt} \langle \Omega(N)a \rangle &= -i\omega \langle \Omega^2(N)a \rangle \\ -\lambda \langle \left(\Omega(N) \left((N+1)f^2(N+1) + Nf^2(N) \right) - 2\Omega(N-1)Nf(N)f(N+1) \right) a \rangle. \end{aligned} \quad (41)$$

These examples show that the equations of motion contain strong nonlinearities introduced by the deformation function f and they do not form a closed system of equations.

4 Equation for the density matrix

Let us rewrite the master equation (27) for the density matrix by means of the number representation. Namely, we take the matrix elements of each term between different number states denoted by $|n\rangle$, and using $N|n\rangle = n|n\rangle$, $a^+|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$, we get

$$\begin{aligned} \frac{d\rho_{mn}}{dt} &= -\frac{i\omega}{2} [mf^2(m) + (m+1)f^2(m+1) - nf^2(n) - (n+1)f^2(n+1)]\rho_{mn} \\ &\quad - [(m+1)f^2(m+1) \left(D_+(\Omega(m)) - \frac{\lambda}{2} \right) + mf^2(m) \left(D_+(\Omega(m-1)) + \frac{\lambda}{2} \right) \\ &\quad + (n+1)f^2(n+1) \left(D_+(\Omega(n)) - \frac{\lambda}{2} \right) + nf^2(n) \left(D_+(\Omega(n-1)) + \frac{\lambda}{2} \right)]\rho_{mn} \\ &\quad + \sqrt{(m+1)(n+1)}f(m+1)f(n+1)[D_+(\Omega(m)) + D_+(\Omega(n)) + \lambda]\rho_{m+1,n+1} \\ &\quad + \sqrt{mn}f(m)f(n)[D_+(\Omega(m-1)) + D_+(\Omega(n-1)) - \lambda]\rho_{m-1,n-1} \end{aligned}$$

$$\begin{aligned}
& -\sqrt{(m+1)n}f(m+1)f^2(n)[D_-(\Omega(m)) + D_-(\Omega(n-1))] \\
& \quad -\frac{i}{\hbar}(D_{pq}(\Omega(m)) + D_{pq}(\Omega(n-1)))\rho_{m+1,n-1} \\
& -\sqrt{m(n+1)}f(m)f(n+1)[D_-(\Omega(m-1)) + D_-(\Omega(n))] \\
& \quad +\frac{i}{\hbar}(D_{pq}(\Omega(m-1)) + D_{pq}(\Omega(n)))\rho_{m-1,n+1} \\
& +\sqrt{(m+1)(m+2)}f(m+1)f(m+2)[D_-(\Omega(m+1)) - \frac{i}{\hbar}D_{pq}(\Omega(m+1))]\rho_{m+2,n} \\
& +\sqrt{(n+1)(n+2)}f(n+1)f(n+2)[D_-(\Omega(n+1)) + \frac{i}{\hbar}D_{pq}(\Omega(n+1))]\rho_{m,n+2} \\
& +\sqrt{m(m-1)}f(m-1)f(m)[D_-(\Omega(m-2)) + \frac{i}{\hbar}D_{pq}(\Omega(m-2))]\rho_{m-2,n} \\
& +\sqrt{n(n-1)}f(n-1)f(n)[D_-(\Omega(n-2)) - \frac{i}{\hbar}D_{pq}(\Omega(n-2))]\rho_{m,n-2}. \quad (42)
\end{aligned}$$

Here we used the abbreviated notation $\rho_{mn} = \langle m|\rho(t)|n \rangle$. This equation, very complicated in form and in indices involved, gives an infinite hierarchy of coupled equations for the matrix elements. When

$$D_-(\Omega(n)) = 0, \quad D_{pq}(\Omega(n)) = 0, \quad (43)$$

the diagonal elements are coupled only amongst themselves and not coupled to the off-diagonal elements. In this case the diagonal elements (populations) satisfy a simpler set of master equations:

$$\begin{aligned}
\frac{dP(n)}{dt} = & -\left((n+1)f^2(n+1)(2D_+(\Omega(n)) - \lambda) + nf^2(n)(2D_+(\Omega(n-1)) + \lambda)\right)P(n) \\
& + (n+1)f^2(n+1)(2D_+(\Omega(n)) + \lambda)P(n+1) + nf^2(n)(2D_+(\Omega(n-1)) - \lambda)P(n-1), \quad (44)
\end{aligned}$$

where we have set $P(n) \equiv \rho_{nn}$. For a q -deformation, Eq. (44) takes the form

$$\begin{aligned}
\frac{dP(n)}{dt} = & -([n+1](2D_+(\Omega(n)) - \lambda) + [n](2D_+(\Omega(n-1)) + \lambda))P(n) \\
& + [n+1](2D_+(\Omega(n)) + \lambda)P(n+1) + [n](2D_+(\Omega(n-1)) - \lambda)P(n-1), \quad (45)
\end{aligned}$$

where, according to Eq. (13),

$$\Omega(n) = \frac{1}{2}([n+2] - [n]). \quad (46)$$

We define the transition rates

$$t_+(n) = (n+1)f^2(n+1)(2D_+(\Omega(n)) - \lambda), \quad t_-(n) = nf^2(n)(2D_+(\Omega(n-1)) + \lambda), \quad (47)$$

which for a q -deformation look like

$$t_+(n) = [n+1](2D_+(\Omega(n)) - \lambda), \quad t_-(n) = [n](2D_+(\Omega(n-1)) + \lambda). \quad (48)$$

With these notations Eq. (44) becomes:

$$\frac{dP(n)}{dt} = t_+(n-1)P(n-1) + t_-(n+1)P(n+1) - (t_+(n) + t_-(n))P(n). \quad (49)$$

The steady state solution of Eq. (49) is found to be

$$P_{ss}(n) = P(0) \prod_{k=1}^n \frac{2D_+(\Omega(k-1)) - \lambda}{2D_+(\Omega(k-1)) + \lambda}. \quad (50)$$

We remark that in the steady state the detailed balance condition holds:

$$t_-(n)P(n) = t_+(n-1)P(n-1). \quad (51)$$

In the particular case of a thermal state, when the diffusion coefficients have the form (28), the stationary solution of Eq. (49) takes the following form:

$$P_{ss}^{th}(n) = Z_f^{-1} \exp\left\{-\frac{\hbar\omega}{2kT} \left((n+1)f^2(n+1) + nf^2(n)\right)\right\}, \quad (52)$$

where

$$Z_f^{-1} = P(0) \exp \frac{\hbar\omega f^2(1)}{2kT} \quad (53)$$

and Z_f is the partition function:

$$Z_f = \sum_{n=0}^{\infty} \exp\left\{-\frac{\hbar\omega}{2kT} \left((n+1)f^2(n+1) + nf^2(n)\right)\right\}. \quad (54)$$

By using the eigenvalues (9), the distribution (52) can be written

$$P_{ss}^{th}(n) = Z_f^{-1} \exp\left(-\frac{E_n}{kT}\right). \quad (55)$$

Expressions (52) and (55) represent the Boltzmann distribution for the deformed harmonic oscillator. In the limit $f \rightarrow 1$ the probability $P_{ss}^{th}(n)$ becomes the usual Boltzmann distribution for the ordinary harmonic oscillator with the well-known partition function $Z = 1/2 \sinh \frac{\hbar\omega}{2kT}$. For the q -oscillator described by Hamiltonian \mathcal{H} in Eq. (8) and weakly coupled to a reservoir kept at the temperature T , the q -deformed partition function can be obtained as a particular case of the partition function Z_f (54), by taking the deformation function (4):

$$Z_q = \sum_{n=0}^{\infty} \exp\left\{-\frac{\hbar\omega}{2kT} \frac{\sinh(\tau(n+1)) + \sinh(\tau n)}{\sinh \tau}\right\}. \quad (56)$$

The results (50) – (56) coincide with those obtained in the previous paper [9], where we considered a linear interaction potential V and the harmonic oscillator operators contained

in this potential are kept undeformed. Therefore, for the equilibrium energy we obtain in the limit of small deformation parameter τ the same expression like that obtained in Ref. [9]:

$$E(t \rightarrow \infty) = \frac{\hbar\omega}{2}(\coth \frac{\hbar\omega}{2kT} + \tau^2 c), \quad (57)$$

where

$$c = \frac{e^\beta}{(e^\beta - 1)^2} \left(\frac{e^\beta + 1}{e^\beta - 1} - \beta \frac{e^{2\beta} + 4e^\beta + 1}{(e^\beta - 1)^2} \right), \quad \beta = \frac{\hbar\omega}{kT}. \quad (58)$$

We note that in the approximation of small deformation, the energy of the damped deformed oscillator depends on the energy $\hbar\omega/2$ of oscillator ground state and on the temperature T . Evidently, when there is no deformation ($\tau \rightarrow 0$), one recovers the energy of the ordinary harmonic oscillator in a thermal bath [13, 14]. In the limit $T \rightarrow 0$, one has $c \rightarrow 0$, $E(\infty) = \hbar\omega/2$ and the deformation does not play any role.

5 Summary and conclusions

Our purpose was to study the dynamics of the deformed quantum harmonic oscillator in a deformed interaction with a dissipative environment, in particular with a thermal bath. We derived in the Born-Markov approximation a master equation for the reduced density operator of the damped f -deformed oscillator. The one-dimensional f - or q -oscillator is a nonlinear quantum oscillator with a specific type of nonlinearity and, consequently, the diffusion and dissipation coefficients which model the influence of the environment on the deformed oscillator strongly depend on the introduced nonlinearities. Compared to the previous paper [9], in the present work the harmonic oscillator operators contained in the interaction terms are deformed. This fact is reflected in the increased degree of the nonlinearity of the equations of motion for the expectation values of observables. In the limit of zero deformation, the master equation takes the form of the master equation for the damped oscillator obtained in the framework of the Lindblad theory of open systems based on quantum dynamical semigroups. We have also derived the equation for the density matrix in the number representation. In the case of a thermal bath we obtained the stationary solution, which is the Boltzmann distribution for the deformed harmonic oscillator. In the steady state the detailed balance condition holds true. In the approximation of a small deformation, we obtained the expression of the equilibrium energy of the deformed harmonic oscillator without deformation of interaction. This energy depends on the oscillator ground state energy and temperature.

The master equation for the damped deformed harmonic oscillator is an operator equation. It could be useful to study its consequences for the density operator by transforming this equation into more familiar forms, such as the partial differential equations of Fokker-Planck type for the Glauber, antinormal ordering and Wigner quasiprobability distributions or for analogous deformed quasiprobabilities [7] associated with the density operator. It could also be interesting to study the properties of the entropy of the damped deformed harmonic oscillator and to find the states which minimize the rate of entropy production for this system. In the case of the undeformed damped oscillator such states are represented by correlated coherent states [29]. We suppose that for the damped deformed oscillator the corresponding states are the deformed (nonlinear) coherent states [15], which play an important role in the description of the phenomenon of environment induced decoherence. The dissipative dynamics of deformed coherent states superposition and the related coherence properties have recently been studied by Mancini and Man'ko [30].

Acknowledgements

One of us (A. I.) is pleased to express his sincere gratitude for the hospitality at the Institut für Theoretische Physik in Giessen. A. I. also gratefully acknowledges the fellowship from DAAD.

References

- [1] P. Raychev, Adv. Q. Chem. **26**, 239 (1995) and references therein
- [2] D. Bonatsos, C. Daskaloyannis, Prog. Part. Nucl. Phys. **43**, 537 (1999) and references therein
- [3] L. C. Biedenharn, J. Phys. A: Math. Gen. **22**, L873 (1989)
- [4] A. J. MacFarlane, J. Phys. A: Math. Gen. **22**, 4581 (1989)
- [5] V. I. Man'ko, G. Marmo, S. Solimeno and F. Zaccaria, Int. J. Mod. Phys. A **8**, 3577 (1993)
- [6] V. I. Man'ko, G. Marmo, S. Solimeno and F. Zaccaria, Phys. Lett. A **176**, 173 (1993)
- [7] P. Aniello, V. I. Man'ko, G. Marmo, S. Solimeno and F. Zaccaria, J. Opt. B: Quant. Semiclass. Opt. **2**, 718 (2000)
- [8] G. Su and M. Ge, Phys. Lett. A **173**, 17 (1993)
- [9] A. Isar and W. Scheid, Physica A **310**, 364 (2002)

- [10] S. Mancini, Phys. Scripta **59**, 195 (1999)
- [11] A. Isar, A. Sandulescu and W. Scheid, Physica A **322**, 233 (2003)
- [12] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976)
- [13] A. Sandulescu and H. Scutaru, Ann. Phys. (N. Y.) **173**, 277 (1987)
- [14] A. Isar, A. Sandulescu, H. Scutaru, E. Stefanescu and W. Scheid, Int. J. Mod. Phys. E **3**, 635 (1994)
- [15] V. I. Man'ko, G. Marmo, E. C. G. Sudarshan and F. Zaccaria, Phys. Scripta **55**, 528 (1997)
- [16] P. P. Kulish and E. V. Damaskinsky, J. Phys. A: Math. Gen. **23**, L415 (1990)
- [17] X. C. Song, J. Phys. A: Math. Gen. **23**, L821 (1990)
- [18] A. P. Polychronakos, Mod. Phys. Lett. A **5**, 2335 (1990)
- [19] T. Curtright and C. Zachos, Phys. Lett. B **243**, 237 (1990)
- [20] R. L. Matos Filho and W. Vogel, Phys. Rev. A **54**, 4560 (1996)
- [21] E. C. G. Sudarshan, Int. J. Theor. Phys. **32**, 1069 (1993)
- [22] G. S. Agarwal and K. Tara, Phys. Rev. A **43**, 492 (1991)
- [23] V. V. Dodonov, Ya. A. Korennoy, V. I. Man'ko and Y. A. Moukhlin, Quant. Semiclass. Opt. **8**, 413 (1996)
- [24] W. H. Louisell, *Quantum Statistical Properties of Radiation*, (Wiley, New York, 1973)
- [25] H. Haken, Rev. Mod. Phys. **47**, 1 (1975)
- [26] G. J. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1982)
- [27] H. Carmichael, *An Open System Approach to Quantum Optics*, Lecture Notes in Physics m18 (Springer, Berlin, 1993)
- [28] D. F. Walls and G. J. Milburn, *Quantum Optics*, (Springer, Berlin, 1995)
- [29] A. Isar, A. Sandulescu and W. Scheid, Phys. Rev. E **60**, 6371 (1999)
- [30] S. Mancini and V. I. Man'ko, Europhys. Lett. **54**, 586 (2001)